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PAIRING OF PARAFERMIONS OF ORDER 2: SENIORITY MODEL

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Abstract

As generalizations of the fermion seniority model, four multi-mode Hamiltonians are considered to investigate some of the consequences of the pairing of parafermions of order two. 2-particle and 4-particle states are explicitly constructed for $H_A \equiv -G\ A^\dagger A$ with $A^\dagger \equiv \frac{1}{2}\sum_{m>0}\ c^\dagger_m c^\dagger_{-m}$ and the distinct $H_C \equiv -G\ C^\dagger C$ with $C^\dagger \equiv \frac{1}{2}\sum_{m>0}\ c^\dagger_{-m} c^\dagger_m$, and for the time-reversal invariant $H_{(-)} \equiv -G\ (A^\dagger - C^\dagger)(A - C)$ and $H_{(+)} \equiv -G\ (A^\dagger + C^\dagger)(A + C)$, which has no analogue in the fermion case. The spectra and degeneracies are compared with those of the usual fermion seniority model.

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1 Introduction

The physics of fermion pairing and fermion condensates [1] is important in contemporary elementary particle physics in precision QCD calculations for hadron spectroscopy (e.g. via lattice gauge theory or chiral effective Lagrangians) and in research on dynamical electro-weak symmetry breaking of the standard model (e.g. via technicolor or a $t\bar{t}$ condensate). In this paper, we construct and study four Hamiltonians as generalizations of the fermion seniority model [2] in order to investigate some of the consequences associated with the pairing of parafermions of order 2. A physical significance of "order $p = 2$" is that 2 or less such parafermions can occur in the same quantum state. Usual fermions correspond to $p = 1$.

Although the idea of the possible existence of fundamental particles associated with other representations of the permutation group is an old and simple one [3], and despite the existence of significant general results in relativistic local quantum field theory concerning properties of elementary particles obeying parastatistics [4,5,6], calculations in this field are sometimes intractable because of algebraic complexities arising from the basic tri-linear commutation relations, see (1-3), versus the standard bi-linear commutation relations which occur in order $p = 1$. Order $p = 2$ is indeed simpler than $p > 2$ because there is a "self-contained set" of 3 relations [4], see (7-9). With respect to representations of the permutation group, consideration of the "order $p = 2$" parafermions is not special in that there are still the two $d = 2$ dimensional representations, a mixed representation still occurs at $d = 3$, and multiple mixed representations still occur at $d = 4$. On the other hand, for $p > 2$, mixed representation(s) with both totally-symmetric and totally-anti-symmetric ones do occur, starting at $d = 3$.

The first Hamiltonian considered in this paper is $H_A \equiv -G\ A^\dagger A$ with $A \equiv \sum_{m>0}\ B^{(m)}$ where

$B^{(m)} = \frac{1}{2} c_{-m} c_m$. The mode index k, l, m ranges from 1 to Ω and $\sum_{m>0}$ denotes summation over $1, 2, ...\Omega$. In this paper, summation symbols are always displayed, so repeated indices are not to be understood to be summed. In many treatments of the usual $p = 1$ seniority model, for instance in nuclear physics applications, Ω is the number of $(l, -l)$ pairs and the "mode index" k, l, m is the magnetic quantum number. In this paper, the time-reversal operation, T, will be analogously defined to exchange $l \leftrightarrow -l$, but except for the use of this exchange operation, no explicit physical significance such as "magnetic quantum number" is associated with the k, l, m index.

In paraquantization, it is instructive to begin by summarizing the parafermi and parabose cases together: The basic commuation relations are

$$[c_k, [c_l^\dagger, c_m]_\mp] = 2\, \delta_{kl}\, c_m, \tag{1}$$

$$[c_k, [c_l^\dagger, c_m^\dagger]_\mp] = 2\, \delta_{kl}\, c_m^\dagger \mp 2\, \delta_{km}\, c_l^\dagger, \tag{2}$$

$$[c_k, [c_l, c_m]_\mp] = 0 \tag{3}$$

following the standard convention that the upper (lower) signs correspond respectively to parafermions (parabosons). The minus subscript is often suppressed so $[A, B] \equiv [A, B]_- \equiv AB - BA$, and $\{A, B\} \equiv [A, B]_+ \equiv AB + BA$. In this paper, the corresponding creation and annihilation operators for the ordinary $p = 1$ fermions are labeled $a_k^\dagger$ and a_l. For $p = 1$, $a_m^\dagger a_{-m}^\dagger = -\, a_{-m}^\dagger a_m^\dagger$, but for $p > 1$, $c_m^\dagger c_{-m}^\dagger$ and $c_{-m}^\dagger c_m^\dagger$ are distinct operators. The number operator for the para-particles is $N_k = \frac{1}{2}[c_k^\dagger, c_k]_\mp \pm \frac{p}{2}$ with the order of the paraparticles, p, being the maximum number of parafermions (parabosons) in a totally symmetric state (anti-symmetric state). We assume a unique vacuum state with the usual properties $c_k|0>= 0$, $<0|0>= 1$, and $c_k c_l^\dagger|0>= p\, \delta_{kl}|0>$.

From here on in this paper, the $c_l^\dagger$, and c_m are parafermi operators of order 2. The following

two commuting operators frequently occur in this pairing analysis:

$$\widehat{N} \equiv \frac{1}{2} \sum_{m>0} \left([c_m^\dagger, c_m] + [c_{-m}^\dagger, c_{-m}] \right) \ + \ 2\,\Omega \tag{4}$$

$$\widehat{\rho} \equiv \frac{1}{2} \sum_{m>0} \left(\{c_m^\dagger, c_m\} - \{c_{-m}^\dagger, c_{-m}\} \right) \tag{5}$$

Note that $\widehat{N}$ is the sum of the parafermion number operators for the 2Ω modes. However, although $c_l^\dagger$, and c_m are parafermi operators of order 2, $\widehat{\rho}$ has the formal structure of being the difference of parabosonic number operators for the $m > 0$ and $m < 0$ modes. The appearance of this T-odd operator $\widehat{\rho}$ is a noteworthy difference versus the ordinary $p = 1$ seniority model, in which it vanishes.

Since $(B^{(m)})^\dagger = \frac{1}{2} c_m^\dagger c_{-m}^\dagger$ and $(D^{(m)})^\dagger = \frac{1}{2} c_{-m}^\dagger c_m^\dagger$ are distinct operators, we also consider a second Hamiltonian $H_C \equiv -G\, C^\dagger C$ with $C \equiv \sum_{m>0}\ D^{(m)}$. In A and in C the parafermions with m and $-m$ are paired and so states constructed as polynomials of A and C will be labeled as states of seniority zero, $s = 0$, since they are states built out of paired particles. Thus, for $H_{A,C}$, the seniority s is the number of unpaired parafermions in the state, just as in the $p = 1$ case.

In the investigation and analysis of fermion pairing phenomena in condensed matter physics, the presence or absence of time reversal invariance and its consequences has been one of the important symmetry considerations [7]. If, in this respect, one does treat the m index on c_m as corresponding to a magnetic quantum number, then under the time reversal operation, $B^{(m)} \leftrightarrow D^{(m)}$ and $H_A \leftrightarrow H_C$. Although there appears to be no obvious violation of time-reversal invariance at the observable's level corresponding to such a discrete switch of Hamiltonians, e.g. spectra will be the same for the two respective Hamiltonians, we find this a somewhat radical formal situation which appears not very easily generalized, in particular with respect to inclusion of kinetic energy terms and perturbations. Accordingly, in this paper we also consider two time-

reversal-invariant Hamiltonians which are mapped into themselves by this time-reversal operation. These are $H_{(-)} \equiv -G\ (A^\dagger - C^\dagger)(A - C)$ and $H_{(+)} \equiv -G\ (A^\dagger + C^\dagger)(A + C)$. Note that $H_{(+)}$ does not exist in the $p = 1$ case. As will be discussed below, results for $H_{(+)}$ such as its spectrum can normally be obtained from those for $H_{(-)}$ by appropriate "substitution rules."

The operators $\widehat{N}$ and $\widehat{\rho}$ commute with $H_{A,C}$ but only $\widehat{N}$, not $\widehat{\rho}$, commutes with $H_{(\mp)}$.

As in the usual quasi-spin formalism [8], Section 2 of this paper treats the algebras associated with the $H_{A,C}$ Hamiltonians, $\widehat{N}$, $\widehat{\rho}$,A, C, and other such two-body operators . Analogous to the $s = 0$ operators A and C, two sets of $s = 2$ two-body operators B_i and D_i with $i = 1, ..., \Omega - 1$ are introduced. In Sections 3 and 4, these two-body operators A, C, B_i and D_i are used to explicitly construct N-particle states with various senorities (N is even). These results are used to study the spectrum for $H_{A,C}$ and for $H_{\mp}$ in comparison with that of the fermion seniority model $H \equiv -G\ \sum_{m>0} a^\dagger_m a^\dagger_{-m} \sum_{l>0} a_{-l} a_l$ which has the $N-$state spectrum

$$E_s(N) = -\frac{1}{4} G(N - s)(2\Omega_{p=1} - N - s + 2); \quad s = 0, 2, ...N. \tag{6}$$

The 2-particle and 4-particle states are explicitly constructed for $H_{A,C}$ in Section 3 and for $H_{(-)}$ in Section 4. In both cases for $s = 4$ the construction of the 4-particle states is only for $\Omega = 4$. For $H_{A,C}$ and for $H_{\mp}$, it is found that the necessary mutual orthogonality properties of the 4-particle states are somewhat awkward to arrange using the two-body operators A, C, B_i and D_i. There are two built-in "parafermi p-saturation" conditions for $p = 2$: $(c^\dagger_k)^3 = 0$ and $(A^\dagger)^M = (C^\dagger)^M = 0$ when $M = \Omega + 1$ [this second fact is also true in the $p = 1$ case]. For some of the additional 4-particle states, for instance, this has the consequence that some state normalization constants vanish for small Ω values, because the states do not then exist. Some results for arbitrary Ω beyond $N = 4$ are derived. In Section 3, in all cases, the spectrum and

degeneracies for $H_{A,C}$ is found to be that of the usual $p = 1$ senority model.

In Section 4, for $H_{(-)}$, for $N = 2, 4$ by explicit construction of orthonormal states, a sizable number of additional states not present in the analysis of H_A are found to occur. For $H_{(-)}$, results are obtained for arbitrary N-particles states which can be constructed as polynomials in only the $A^\dagger$ and $C^\dagger$ operators. In all cases, the spectrum of $H_{(-)}$ is found to be that of the $p = 1$ senority model, except that $\Omega_{p=1}$ in (6) is replaced by 2Ω. However, for $H_{(-)}$ there are many additional degeneracies beyond those of the usual $p = 1$ senority model. These degeneracies can be specified by an appropriate use of the seniority number, s.

A primary motivation for studying the seniority model is because it is a simple model which has been used for fermions to theoretically investigate and exhibit consequences of fermion-pairing, of the microscopic realization of superconductivity, and thereby of spontaneous symmetry breaking. The most surprising result of this paper's analysis is that in this multi-mode framework for parafermions of order two, it is indeed possible to algebraically investigate the spectrum for each of the four Hamiltonians. In hindsight, this tractability is partially a consequence of three facts: (i) in a single mode, the pairing-operators $B^{(m)} \equiv \frac{1}{2} c_{-m} c_m$ and $D^{(m)} \equiv \frac{1}{2} c_m c_{-m}$ separately lead to a two-body operator, quasi-spin Hamiltonian structure which is similar to that of the fermionic case, (ii) for different modes, six Hermitian pairing-operators mutually commute: These operators are ${B_1}^{(m)} \equiv \frac{1}{2}({B^{(m)}}^\dagger + B^{(m)})$, ${B_2}^{(m)} \equiv \frac{i}{2}({B^{(m)}}^\dagger - B^{(m)})$, and ${B_3}^{(m)} \equiv \frac{1}{2}[{B^{(m)}}^\dagger, B^{(m)}]$ and analogously for ${D_a}^{(m)}$, and $[B_a^{(l)}, B_b^{(m)}] = [D_a^{(l)}, D_b^{(m)}] = [B_a^{(l)}, D_b^{(m)}] = 0$ for $l \neq m$ where $a, b = 1, 2, 3$, and (iii) for these operators in the same mode also $[B_a^{(m)}, D_b^{(m)}] = 0$.

2 Two-body operator algebras

For parafermions of order 2, one has the relations

$$c_k^\dagger c_l c_m + c_m c_l c_k^\dagger = 2\ \delta_{kl}\ c_m \tag{7}$$

$$c_k c_l^\dagger c_m + c_m c_l^\dagger c_k = 2\ \delta_{kl}\ c_m + 2\ \delta_{lm}\ c_k \tag{8}$$

$$c_k c_l c_m + c_m c_l c_k = 0 \tag{9}$$

plus the Hermitian conjugate relations. Note from the left-hand side of these relations that there is a simple left↔right reordering symmetry. On the vacuum state $c_k c_l^\dagger |0>= 2\delta_{kl}|0>$. A consequence of (9) is the "parafermi p-saturation" conditions noted in the "Introduction". Useful commutators involving $c_k^\dagger$, $c_{-k}^\dagger$, ... pairs are in the appendix of this paper.

It follows that the $s=0$ two-body operator $A \equiv \sum_{m>0}\ B^{(m)} = \frac{1}{2}\sum_{m>0} c_{-m}c_m$ has the quasi-spin algebraic relations:

$$[A, A^\dagger] = -2Z_{A3}, \quad [Z_{A3}, A^\dagger] = A^\dagger, \quad [Z_{A3}, A] = -A \tag{10}$$

where $A^\dagger \equiv A_1 + iA_2$ with $i = \sqrt{-1}$, $A_3 \equiv Z_{A3}$. For $\overrightarrow{S_A}^2 \equiv (A_1)^2 + (A_2)^2 + (Z_{A3})^2$, one finds $[\overrightarrow{S_A}^2, A_{1,2,3}] = 0$ and $H_A = -G(\overrightarrow{S_A}^2 - (Z_{A3})^2 + Z_{A3})$. Since $[A^\dagger A, Z_{A3}] = 0$, H_A and Z_{A3} can be simultaneously diagonalized. The H_A eigenvalues are $E_A = -G\{s_A(s_A+1) - (z_{3A}\ )^2 + z_{3A}\}$. The explicit N-particle parafermion states of various seniorities corresponding to this spectrum are constructed in Section 3 below. In terms of the operators $\widehat{N}$ and $\widehat{\rho}$ defined by (4,5)

$$Z_{A3} \equiv \frac{1}{4}(\widehat{N} - \widehat{\rho} - 2\Omega) \tag{11}$$

Similarly, for $C = \sum_{m>0}\ D^{(m)} = \frac{1}{2}\sum_{m>0} c_m c_{-m}$,

$$[C, C^\dagger] = -2Y_{C3}, \quad [Y_{C3}, C^\dagger] = C^\dagger = C_1 + iC_2, \quad [Y_{C3}, C] = -C \tag{12}$$

where

$$Y_{C3} \equiv C_3 \equiv \frac{1}{4}(\widehat{N} + \widehat{\rho} - 2\Omega), \tag{13}$$

Since $[C^\dagger C, Y_{C3}] = 0$, H_C and Y_{C3} can be simultaneously diagonalized. The H_C eigenvalues are $E_A = -G\{s_C(s_C + 1) - (y_{3C}\ )^2 + y_{3C}\}$.

Useful eigenvalues for the vacuum state are

$$AA^\dagger|0>= CC^\dagger|0>= \Omega|0>, Z_{A3}|0>= Y_{C3}|0>= -\frac{1}{2}\Omega|0> \tag{14}$$

and $\widehat{N}|0>= \widehat{\rho}|0>= 0$. Unlike in the $p = 1$ case, here for the $s = 0$ N-particle states due to the occurrence of $\widehat{\rho}$ as well as $\widehat{N}$, while $\widehat{N}(A^\dagger)^M|0>= 2M(A^\dagger)^M|0>$ and $\widehat{N}(C^\dagger)^M|0>= 2M(C^\dagger)^M|0>$ for M a non-negative integer, there is a minus sign in $\widehat{\rho}(C^\dagger)^M|0>= -2M(C^\dagger)^M|0>$ versus $\widehat{\rho}(A^\dagger)^M|0>= 2M(A^\dagger)^M|0>$. These states have respectively the energy eigenvalues $E_0^{(A,C)}(2M) = -GM(\Omega - M + 1)$.

There are the useful commutators

$$\begin{aligned} [\widehat{N}, A^\dagger] &= 2A^\dagger, \quad [\widehat{\rho}, A^\dagger] = -2A^\dagger && (15)\\ [H_A, A^\dagger] &= -\frac{1}{2}G(\widehat{\rho} - \widehat{N} + 4 + 2\Omega)A^\dagger && (16)\\ &= -\frac{1}{2}GA^\dagger(\widehat{\rho} - \widehat{N} + 2\Omega) && (17) \end{aligned}$$

The $\widehat{\rho}$ again appears with an extra minus sign in the analogous expressions

$$\begin{aligned} [\widehat{N}, C^\dagger] &= 2C^\dagger, \quad [\widehat{\rho}, C^\dagger] = 2C^\dagger && (18)\\ [H_C, C^\dagger] &= -\frac{1}{2}G(-\widehat{\rho} - \widehat{N} + 4 + 2\Omega)C^\dagger && (19)\\ &= -\frac{1}{2}GC^\dagger(-\widehat{\rho} - \widehat{N} + 2\Omega) && (20) \end{aligned}$$

Note $[H_A, C^\dagger] = [H_C, A^\dagger] = 0$.

To explicitly construct states with seniority $s \neq 0$, we define the $s = 2$ two-body operators

$$
\begin{aligned}
B_i^\dagger &\equiv \frac{1}{2}\left(\sum_{j=1}^{i} c_j^\dagger c_{-j}^\dagger - i c_{(i+1)}^\dagger c_{-(i+1)}^\dagger\right) && (21)\\
&\equiv \sum_{m=1}^{i} B^{(m)\dagger} - i B^{(i+1)\dagger}; \quad i = 1, ..., (\Omega - 1) && (22)
\end{aligned}
$$

and

$$
D_i^\dagger \equiv \frac{1}{2}\left(\sum_{j=1}^{i} c_{-j}^\dagger c_j^\dagger - i c_{-(i+1)}^\dagger c_{(i+1)}^\dagger\right); i = 1, ..., (\Omega - 1) \tag{23}
$$

The 2-particle states $B_i^\dagger|0>$ and $D_i^\dagger|0>$ with $s = 2$ respectively have zero $H_{A,C}$ energy eigenvalues, $\widehat{N}$ eigenvalues of 2, and respectively $\widehat{\rho}$ eigenvalues of ∓ 2 like respectively $A^\dagger, C^\dagger$. For Ω arbitrary,

$$
[A, B_i^\dagger] = -2Z_{3Bi}, \quad [A^\dagger, B_i^\dagger] = [B_i^\dagger, B_j^\dagger] = 0 \tag{24}
$$

$$
[H_A, B_i^\dagger] = 2GA^\dagger Z_{3Bi}, \quad [\widehat{N}, B_i^\dagger] = 2B_i^\dagger, \quad [\widehat{\rho}, B_i^\dagger] = -2B_i^\dagger \tag{25}
$$

where

$$
Z_{3Bi} \equiv \frac{1}{4}\left(\sum_{j=1}^{i} \{c_j^\dagger c_j - c_{-j} c_{-j}^\dagger\} - i\{c_{(i+1)}^\dagger c_{(i+1)} - c_{-(i+1)} c_{-(i+1)}^\dagger\}\right) \tag{26}
$$

and

$$
\begin{aligned}
[Z_{3Bi}, A^\dagger] &= B_i^\dagger, \quad [Z_{3A}, B_i^\dagger] = B_i^\dagger && (27)\\
[Z_{3A}, Z_{3Bi}] &= [Z_{3Bi}, Z_{3Bj}] = 0 && (28)
\end{aligned}
$$

On the vacuum state, $Z_{3Bi}|0>= 0$, so $AB_i^\dagger|0>= B_i A^\dagger|0>= 0$. Alternatively, in terms of mode operators

$$
\begin{aligned}
Z_{3Bi} &\equiv \frac{1}{4}\left(\sum_{m=1}^{i} \{\widehat{N}_B^{(m)} - \widehat{\rho}_B^{(m)}\} - i\{\widehat{N}_B^{(i+1)} - \widehat{\rho}_B^{(i+1)}\}\right) && (29)\\
&= \sum_{m=1}^{i} Z_3^{(m)} - i Z_3^{(i+1)} && (30)
\end{aligned}
$$

where

$$\widehat{N}_B^{(m)} \equiv \frac{1}{2}\left([c_m^\dagger, c_m] + [c_{-m}^\dagger, c_{-m}]\right) + 2\,\Omega \tag{31}$$

$$\widehat{\rho}_B^{(m)} \equiv \frac{1}{2}\left(\{c_m^\dagger, c_m\} - \{c_{-m}^\dagger, c_{-m}\}\right) \tag{32}$$

$$Z_3^{(m)} \equiv \frac{1}{4}\left(c_m^\dagger c_m - c_{-m} c_{-m}^\dagger\right) \tag{33}$$

For $\Omega > 2$, these operators do not completely close at the level of the A's,C's,B_i's,D_i's but instead involve mode operators:

$$[Z_{3Bi}, B_j^\dagger] = \left\{ \begin{array}{c} \sum_{m=1}^{i} B^{(m)\dagger} + (i)^2 B^{(i+1)\dagger}, i = j \\ \\ B_{i<}^\dagger, \quad i \neq j \end{array} \right\} \tag{34}$$

where $i<$ denotes the smaller of i, j.

Similarly,

$$[C, D_i^\dagger] = -2Y_{3Di}, \quad [C^\dagger, D_i^\dagger] = [D_i^\dagger, D_j^\dagger] = 0 \tag{35}$$

$$[H_C, D_i^\dagger] = 2GC^\dagger Y_{3Di}, \quad [\widehat{N}, D_i^\dagger] = 2D_i^\dagger, \quad [\widehat{\rho}, D_i^\dagger] = 2D_i^\dagger \tag{36}$$

where

$$Y_{3Di} \equiv \frac{1}{4}(\sum_{j=1}^{i}\{c_{-j}^\dagger c_{-j} - c_j c_j^\dagger\} - i\{c_{-(i+1)}^\dagger c_{-(i+1)} - c_{(i+1)} c_{(i+1)}^\dagger\}) \tag{37}$$

$$= \sum_{m=1}^{i} Y_3^{(m)} - iY_3^{(i+1)} \tag{38}$$

and

$$[Y_{3Di}, C^\dagger] = D_i^\dagger, \quad [Y_{3C}, D_i^\dagger] = D_i^\dagger \tag{39}$$

$$[Y_{3C}, Y_{3Di}] = [Y_{3Di}, Y_{3Dj}] = 0 \tag{40}$$

$$[Y_{3Di}, D_j^\dagger] = \left\{ \begin{array}{c} \sum_{m=1}^{i} D^{(m)\dagger} + (i)^2 D^{(i+1)\dagger}, i = j \\ \\ D_{i<}^\dagger, \quad i \neq j \end{array} \right\} \tag{41}$$

On the vacuum state $Y_{3Di}|0>=0$, so $CD_i^\dagger|0>= D_iC^\dagger|0>=0$, and

$$Y_{3Di} \equiv \frac{1}{4}(\sum_{m=1}^{i}\{\widehat{N}_B^{(m)}+\widehat{\rho}_B^{(m)}\} - i\{\widehat{N}_B^{(i+1)}+\widehat{\rho}_B^{(i+1)}\}) \tag{42}$$

$$= \sum_{m=1}^{i} Y_3^{(m)} - iY_3^{(i+1)} \tag{43}$$

where $Y_3^{(m)} \equiv \frac{1}{4}\left(c_{-m}^\dagger c_{-m} - c_m c_m^\dagger\right)$. The states $B_i^\dagger|0>, A^\dagger|0>$ have Z_{A3} eigenvalues of $(1-\frac{1}{2}\Omega)$, and Y_{C3} eigenvalues of $(-\frac{1}{2}\Omega)$; similarly $D_i^\dagger|0>, C^\dagger|0>$ have Y_{C3} eigenvalues of $(1-\frac{1}{2}\Omega)$, and Z_{A3} eigenvalues of $(-\frac{1}{2}\Omega)$.

Note that

$$[B_i, B_j^\dagger] = -2Z_{3Bi<} - \frac{1}{2}\delta_{ij}i(i+1)\{c_{(i+1)}^\dagger c_{(i+1)} - c_{-(i+1)}c_{-(i+1)}^\dagger\} \tag{44}$$

where again $i<$ denotes the smaller of i,j; the last term's factor also appears in (26). On the vacuum state $B_iB_j^\dagger|0>= i(i+1)\delta_{ij}||0>$, so $B_i^\dagger|0>$ are orthogonal for different i values. Similarly,

$$[D_i, D_j^\dagger] = -2Y_{3Di<} - \frac{1}{2}\delta_{ij}i(i+1)\{c_{-(i+1)}^\dagger c_{-(i+1)} - c_{(i+1)}c_{(i+1)}^\dagger\} \tag{45}$$

and $D_iD_j^\dagger|0>= i(i+1)\delta_{ij}||0>$.

3 Spectrum of H_A

Useful relations for treating arbitrary N particle states include: for $r=1,2,...$

$$A(A^\dagger)^r|0> = r(\Omega - r + 1)(A^\dagger)^{r-1}|0> \tag{46}$$

$$(A)^r(A^\dagger)^r|0> = r!\Omega(\Omega-1)\cdots(\Omega-r+1)|0> \tag{47}$$

$$Z_{A3}(A^\dagger)^r|0> = (r - \frac{1}{2}\Omega)(A^\dagger)^r|0> \tag{48}$$

Since we find the H_A case to be relatively simple, for instance it has the same spectrum as in the usual $p=1$ senority model, we do not evaluate normalization constants in this section.

For arbitrary $N \geq 2$, the $s = 0$ states $|N^A >_0= (A^\dagger)^{N/2}|0>$ have energy eigenvalues $E_0^{(A)}(N) = -\frac{1}{4}GN(2\Omega - N + 2)$.

For the $s = 2$, N-particle states with $N \geq 2$, $|N^A >_2 \equiv B_i^\dagger (A^\dagger)^{(N-2)/2}|0>$ have $E_2^{(A)}(N) = -\frac{1}{4}G(N-2)(2\Omega - N)$.

For $s = 4$, we are interested in both the H_A states for themselves and also for comparison below with those occurring in the analysis of the $H_{(-)}$ states: For $\Omega = 4$, the explicit orthogonal 4-particle states include two with zero H_A eigenvalues. From the analogous $p = 1$ spectra and using completeness, we classify them as $s = 4$ states. These two states are $|4_a^A >_4= \frac{2}{3}B_1^\dagger(B_3^\dagger - B_2^\dagger)|0>$, $|4_b^A >_4= \frac{1}{\sqrt{3}}B_2^\dagger(B_3^\dagger + A^\dagger)|0>$; to achieve orthogonality these states are somewhat complicated in terms of the $B_i^\dagger$ operators. The other $\Omega = 4$ states are special cases of ones discussed above: $|4^A >_0= (A^\dagger)^2|0>$, $|4_i^A >_2= A^\dagger B_i^\dagger|0>$ where $i = 1, 2, 3$.

Next, for the $s = 4$, N-particle states with $N \geq 4$, we consider $|N^A >_4 \equiv B_i^\dagger B_j^\dagger A^\dagger)^{(N-4)/2}|0>$ with $i < j$. Note that it is at the $s = 4$ senority level that in using the $B_i^\dagger$ operators, the orthogonality requirements started to produce complications for the H_A case, and similarly in the $H_{(-)}$ case below. So in considering only $|N^A >_4$, we are ignoring this difficulty. When H_A operates on this state, one can commute A past the first $B_i^\dagger$ using $[A, B_i^\dagger] = -2Z_{3Bi}$. In commuting Z_{3Bi} past the next $B_j^\dagger$ one can use (34), $[Z_{3Bi}, B_j^\dagger] = B_i^\dagger$ since $i < j$, however, $B_i^\dagger$ produces an $(N-2)$-particle state. We proceed by dropping such terms because such $(N-2)$ terms will not contribute if the H_A expectation value is calculated. In this manner, we obtain $E_4^{(A)}(N) = -\frac{1}{4}G(N-4)(2\Omega - N - 2)$. This is not a complete derivation because the $|N^A >_4$ are not mutually orthogonal. If we proceed similarly, for arbitrary senority $s = 2t$, then for N-particle states with $N \geq s$, $|N^A >_s \equiv \{B_{i_1}^\dagger B_{1_2}^\dagger \cdots B_{i_t}^\dagger\}(A^\dagger)^{(N-s)/2}|0>$ with $i_1 < i_2 < \cdots < i_t$, we obtain

$E_s^{(A)}(N) = -\frac{1}{4}G(N-s)(2\Omega - N - s + 2); \quad s = 0, 2, ...N$ by induction.

4 Analysis of $H_{(-)}$

For the Hamiltonians $H_{(\mp)}$ whereas $[\widehat{N}, H_{(\mp)}] = 0$,

$$[\widehat{\rho}, H_{(\mp)}] = \mp 4G(A^\dagger C - C^\dagger A) = \mp 4(H_{AC} - H_{CA}) \tag{49}$$

so the N-particle eigenstates of $H_{(\mp)}$ can no longer be classified by the eigenvalues of $\widehat{\rho}$. Note that $H_{(\mp)} = H_A \pm H_{AC} \pm H_{CA} + H_C$ where $H_{AC} = GA^\dagger C^\dagger$ and $H_{CA} = GC^\dagger A^\dagger$.

The algebra associated with $H_{(-)}$ includes the equations

$$[A \pm C, A^\dagger \pm C^\dagger] = -N_{AC3}, \quad [A + C, A^\dagger - C^\dagger] = \widehat{\rho} \tag{50}$$

$$[\widehat{N}, A^\dagger \mp C^\dagger] = 2(A^\dagger \mp C^\dagger), \quad [\widehat{\rho}, A^\dagger \mp C^\dagger] = -2(A^\dagger \pm C^\dagger) \tag{51}$$

where $N_{AC3} = (\widehat{N} - 2\Omega) = 2Z_{A3} + 2Y_{C3} = \frac{1}{2}\sum_{m>0}\left([c_m^\dagger, c_m] + [c_{-m}^\dagger, c_{-m}]\right)$ includes the zero point energy. In comparison, note that $\widehat{\rho} = -2Z_{A3} + 2Y_{C3}$. Also

$$[H_{(-)}, A^\dagger - C^\dagger] = G(A^\dagger - C^\dagger)N_{AC3} \tag{52}$$

$$[H_{(-)}, A^\dagger + C^\dagger] = -G(A^\dagger - C^\dagger)\widehat{\rho} \tag{53}$$

$N = 2, 4$ **particle states:**

We list the orthogonal 2-particle states, their energy eigenvalues, and normalization constants $N_s(N) \equiv {}_s < N|N >_s$:

Ones with $s = 0$,

$|2^- >_0 = (A^\dagger - C^\dagger)|0 >$, $E_0^{(-)}(2) = -2G\Omega$; $N_0(2) = 2\Omega$

$|\breve{2}^- >_0 = (A^\dagger + C^\dagger)|0 >$, $E_0^{(-)}(\breve{2}) = 0$; $N_0(\breve{2}) = 2\Omega$

Ones with $s = 2$,

$|2_i^- >_2 = (B_i^\dagger - D_i^\dagger)|0>$, $E_2^{(-)}(2) = 0$; $N_2(2) = 2i(i+1)$

$|\breve{2}_i^- >_2 = (B_i^\dagger + D_i^\dagger)|0>$, $E_2^{(-)}(\breve{2}) = 0$; $N_2(\breve{2}) = 2i(i+1)$

In calculating degeneracies, it is to be understood that $i, j = 1, 2, ..., \Omega - 1$.

The additional states not present in the $H_{A,C}$ cases are denoted with a "breve", or "short vowel", accent. Such states are therefore not analogous to ones in the usual $p = 1$ seniority model.

We similarly list the orthogonal 4-particle states:

Ones with $s = 0$,

$|4^- >_0 = (A^\dagger - C^\dagger)^2|0>$, $E_0^{(-)}(4) = -2G(2\Omega - 1)$; $N_0(4) = 2\Omega(4\Omega - 1)$

$|\breve{4}^- >_0 = (A^\dagger - C^\dagger)(A^\dagger + C^\dagger)|0>$, $E_0^{(-)}(\breve{4}) = -2G(\Omega - 1)$; $N_0(\breve{4}) = 2(\Omega)^2$

$|\breve{4}_1^- >_0 = (\{A^\dagger\}^2 + \{C^\dagger\}^2 + 2(\frac{\Omega-1}{\Omega})A^\dagger C^\dagger)|0>$, $E_0^{(-)}(\breve{4}_1) = 0$; $N_0(\breve{4}_1) = 4(2\Omega - 1)(\Omega - 1)$

For instance in $|\breve{4}_1^- >_0$, a "number subscript" is used on the additional states label to denote ones constructed with an Ω dependence so as to achieve orthogonality. Due to the parafermi saturation such a state is absent for Ω sufficiently small; this is seen in the norm vanishing and in the vanishing of a factor like $(\frac{\Omega-1}{\Omega})$. Completeness in each "$N, s$ sector" is achieved by construction.

Ones with $s = 2$, which are orthogonal for $i \neq j$,

$|4_i^- >_2 = (B_i^\dagger - D_i^\dagger)(A^\dagger - C^\dagger)|0>$, $E_2^{(-)}(4) = -2G(2\Omega - 2)$; $N_2(4) = 4(\Omega - 1)i(i+1)$

$|\breve{4}_i^- >_2 = (B_i^\dagger + D_i^\dagger)(A^\dagger - C^\dagger)|0>$, $E_2^{(-)}(\breve{4}) = -2G(2\Omega - 2)$; $N_2(\breve{4}) = 4(\Omega - 1)i(i+1)$

$|\breve{4}_{i1}^- >_2 = (\Omega\{B_i^\dagger A^\dagger + D_i^\dagger C^\dagger\} + (\Omega - 2)\{B_i^\dagger C^\dagger + D_i^\dagger A^\dagger\})|0>$, $E_2^{(-)}(\breve{4}_1) = 0$; $N_2(\breve{4}_{i1}) = 4\Omega(\Omega - 1)(\Omega - 2)i(i+1)$

$|\breve{4}_{i2}^- >_2 = (\Omega\{B_i^\dagger A^\dagger - D_i^\dagger C^\dagger\} + (\Omega - 2)\{B_i^\dagger C^\dagger - D_i^\dagger A^\dagger\})|0>$, $E_2^{(-)}(\breve{4}_2) = 0$; $N_2(\breve{4}_{i2}) = 4\Omega(\Omega -$

$1)(\Omega-2)i(i+1)$

Ones with $s=4$, for the case $\Omega=4$; all with zero energy eigenvalues and the same normalization constant $N_4(4)=144$,

$|4^- >_4 = (B_1^\dagger - D_1^\dagger)(B_3^\dagger - B_2^\dagger - \{D_3^\dagger - D_2^\dagger\})|0>;$

$|\breve{4}_a^- >_4 = (B_1^\dagger - D_1^\dagger)(B_3^\dagger - B_2^\dagger + \{D_3^\dagger - D_2^\dagger\})|0>;$

$|\breve{4}_b^- >_4 = (B_1^\dagger + D_1^\dagger)(B_3^\dagger - B_2^\dagger - \{D_3^\dagger - D_2^\dagger\})|0>;$

$|\breve{4}_c^- >_4 = (B_1^\dagger + D_1^\dagger)(B_3^\dagger - B_2^\dagger + \{D_3^\dagger - D_2^\dagger\})|0>;$

Plus two analogues of H_A states, and two additional states,

$|4_\alpha^- >_4 = (B_2^\dagger\{B_3^\dagger + A^\dagger\} + D_2^\dagger\{D_3^\dagger + C^\dagger\})|0>;\ N_4(4_\alpha)=336$

$|\breve{4}_\alpha^- >_4 = (B_2^\dagger\{B_3^\dagger + A^\dagger\} - D_2^\dagger\{D_3^\dagger + C^\dagger\})|0>;\ N_4(\breve{4}_\alpha)=336$

$|4_\beta^- >_4 = \frac{1}{\sqrt{2}}(B_2^\dagger D_3^\dagger + D_2^\dagger B_3^\dagger)|0>;\ N_4(4_\beta)=144$

$|\breve{4}_\beta^- >_4 = \frac{1}{\sqrt{2}}(B_2^\dagger D_3^\dagger - D_2^\dagger B_3^\dagger)|0>;\ N_4(\breve{4}_\beta)=144$

Some N-particle states constructed as polynomials in $A^\dagger$ and $C^\dagger$:

Useful relations for treating arbitrary N particle states include: for $r=1,2,...$

$$(A-C)(A^\dagger - C^\dagger)^r|0> = r(2\Omega - r + 1)(A^\dagger - C^\dagger)^{r-1}|0> \tag{54}$$

$$(A\mp C)^r(A^\dagger \mp C^\dagger)^r|0> = r!\sum_{t=0}^{r}\binom{r}{t}\Omega(\Omega-1)\cdots(\Omega-r+t+1)\,\Omega(\Omega-1)\cdots(\Omega-t+1)|0>$$

and

$$\widehat{N}(A^\dagger)^{r_1}(C^\dagger)^{r_2}|0> = 2(r_1+r_2)(A^\dagger)^{r_1}(C^\dagger)^{r_2}|0> \tag{55}$$

$$\widehat{\rho}(A^\dagger)^{r_1}(C^\dagger)^{r_2}|0> = -2(r_1-r_2)(A^\dagger)^{r_1}(C^\dagger)^{r_2}|0> \tag{56}$$

Ignoring the mutual orthogonality requirement, for arbitrary N, the following seniority $s_{p=1} = 0$ states are found to have the following associated eigenvalues:

For $N \geq 2$, $(A^\dagger - C^\dagger)^{N/2}|0>$ has $E_0^{(-)}(N) = -\frac{1}{4}GN(4\Omega - N + 2)$.

For $\breve{N} \geq 2$, $(A^\dagger - C^\dagger)^{(N-2)/2}(A^\dagger + C^\dagger)|0>$ has $E_0^{(-)}(\breve{N}) = -\frac{1}{4}G(N-2)(4\Omega - N)$.

For $\breve{N}' \geq 4$, $(A^\dagger - C^\dagger)(A^\dagger + C^\dagger)^{(N-2)/2}|0>$ also has $E_0^{(-)}(\breve{N}') = -\frac{1}{4}G(N-2)(4\Omega - N)$. So for $N \geq 4$, both these states are degenerate with the $s = 2$ states with N-particles.

For $\breve{N}'' \geq 4$, $(A^\dagger - C^\dagger)^{(N-4)/2}(\{A^\dagger\}^2 + \{C^\dagger\}^2 + 2\{\frac{\Omega-1}{\Omega}\}A^\dagger C^\dagger)|0>$ has $E_0^{(-)}(\breve{N}'') = -\frac{1}{4}G(N-4)(4\Omega - N - 2)$ so this state is degenerate with the $s = 4$ state with N-particles.

For these states, ignoring orthogonality, the $H_{(-)}$ spectrum is found to be the same as that of H_A, except that $\Omega_{p=1}$ in (6) is replaced by 2Ω. There are, however, many additional degeneracies which can be counted via completeness in each the $N, s_{p=1}$ sector, for Ω sufficiently large so that the absence of states due to p-saturation can be ignored.

The states of the $H_{(+)}$ spectrum follow isomorphically by letting $C \rightarrow -C$ and $D_i^\dagger \rightarrow -D_i^\dagger$: for instance, $|2^+>_0 = (A^\dagger + C^\dagger)|0>$ has $E_0^{(+)}(2) = -2G\Omega$; $|\breve{2}^+>_0 = (A^\dagger - C^\dagger)|0>$ has $E_0^{(+)}(\breve{2}) = 0$; $|2_i^+>_2 = (B_i^\dagger + D_i^\dagger)|0>$ has $E_2^{(+)}(2) = 0$; and $|\breve{2}_i^+>_2 = (B_i^\dagger - D_i^\dagger)|0>$ has $E_2^{(+)}(\breve{2}) = 0$.

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Appendix: ${c_k}^\dagger$, ${c_{-k}}^\dagger$,... pair commutators

Although equations (7-9) are sufficient, the following commutators are algebraically sometimes

more direct or useful as checks:

$$[c_k c_{-k}, c_l^\dagger c_{-l}^\dagger] \;=\; 2\,\delta_{-k,l}(c_k c_k^\dagger - c_{-k}^\dagger\, c_{-k}) \qquad (57)$$

$$[c_k^\dagger c_{-k}^\dagger, c_l^\dagger c_{-l}^\dagger] \;=\; 0 \qquad (58)$$

From the latter it follows that $[B^{(m)}, B^{(l)}] = [D^{(m)}, D^{(l)}] = [B^{(m)}, D^{(l)}] = 0$, so $[A, C] = [B_i, B_j] = [D_i, D_j] = [B_i, D_j] = 0$.

$$[c_k^\dagger c_k, c_l^\dagger c_{-l}^\dagger] \;=\; 2\,\delta_{k,l} c_k^\dagger\, c_{-k}, \quad [c_k c_k^\dagger, c_l^\dagger c_{-l}^\dagger] = -2\,\delta_{k,-l} c_{-k}^\dagger\, c_k \qquad (59)$$

$$[c_k^\dagger c_k, c_l^\dagger c_l] \;=\; [c_k^\dagger c_k, c_l c_l^\dagger] = 0 \qquad (60)$$

The mode operators $B^{(m)}$ also satisfy $[B^{(m)}, B^{(m)\dagger}] = -2Z_3^{(m)}$, $[Z_3^{(m)}, B^{(m)\dagger}] = B^{(m)\dagger}$, where $B^{(m)\dagger} \equiv B_1^{(m)} + iB_2^{(m)}$ and $B_3^{(m)} \equiv Z_3^{(m)}$. See also (31-33). Also, $[D^{(m)}, D^{(m)\dagger}] = -2Y_3^{(m)}$, $[Y_3^{(m)}, D^{(m)\dagger}] = D^{(m)\dagger}$. Thus, for both the $B^{(m)}$'s and $D^{(m)}$'s one has a two-body operator, quasi-spin, and single-mode Hamiltonian structure, analogous to those at the A's and C's level.

Some of the commutators vanish at the A's and C's level because $[B_a^{(l)}, B_b^{(m)}]|_{l\neq m} = [D_a^{(l)}, D_b^{(m)}]|_{l\neq m} = [B_a^{(l)}, D_b^{(m)}] = 0$ for $a, b = 1, 2, 3$. Thus, $[A, C^\dagger] = [B_i, D_i^\dagger] = [A, D_i^\dagger] = [A, D_i] = [C, B_i^\dagger] = [C, B_i] = 0$, and $[Z_{A3}, Y_{C3}] = [Z_{3Bi}, Y_{C3}] = [Z_{A3}, Y_{3Di}] = [Z_{3Bi}, Y_{3Di}] = 0$. Also, $[Z_{A3}, C^\dagger] = [Z_{3Bi}, C^\dagger] = [Z_{A3}, D_i^\dagger] = [Z_{3Bi}, D_i^\dagger] = 0$, $[Y_{C3}, A^\dagger] = [Y_{3Di}, A^\dagger] = [Y_{C3}, B_i^\dagger] = [Y_{3Di}, B_i^\dagger] = 0$, and their adjoints vanish.

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